Classical Mechanics and Statistical/Thermodynamics

January 2022

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\,e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\,e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i a x - b x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^{n+1}\frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$

$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$

$\mu_0 = 4\pi \times 10^{-7}$T m/A

electronic charge $e = 1.60 \times 10^{-19}$C

electronic mass $m_e = 9.11 \times 10^{-31}$kg

Density of pure water: 1.00gm/cm^3.

Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K

Planck's constant: $\hbar = 6.63 \times 10^{-34}\mathrm{m}^2\mathrm{kg/s}$

speed of light: $c = 3.00 \times 10^8$m/s

Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

Classical Mechanics

1. A uniform rod of length L and mass M has a bead of mass m glued a distance $L/4$ from its end as shown. The rod is supported by a thin horizontal, frictionless pin acting as an axle, and passing through the rod's center of mass. The system starts at rest, with the the rod placed as shown, and gravity pointing down (down the page). The radius of the bead is small enough that you can consider it to be a point mass located a distance $L/2$ from the axle.

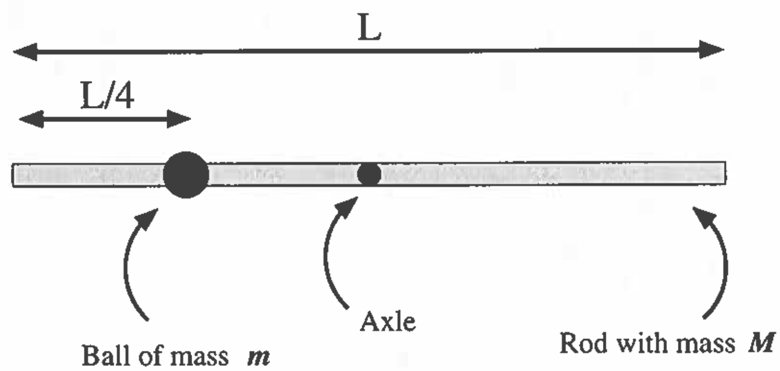


Figure 1: The rod rotates without friction.

(a) Calculate the moment of inertia of the rod around its center of mass. (1 point)

(b) If the system is released from rest so that it may rotate freely about the axle, what is its initial angular acceleration? (2 points)

(c) Calculate the angular velocity of the rod when the bead reaches its lowest point. (2 points)

(d) At what point between its initial release and its lowest point on the arc is the *total linear acceleration* of the bead a maximum? (3 points)

(e) When the ball is at its lowest point on the arc the glue fails and the bead slides to the end of the rod in a very short time, which we will treat as instantaneous. The bead does not fall off the end, but stops at a distance $L/2$ from the axle. When the ball reaches the end of the rod, will the angular velocity of the rod increase, decrease, or remain the same? Explain your answer. (1 point)

(f) In the situation described in part (e) above, will the total kinetic energy of the system increase, decrease, or remain the same? Explain your answer. (1 point)

2. A particle of mass, m moves without friction, confined to a surface given by $z = \frac{1}{2}ar^2$, where $a > 0$ is a constant. A gravitational field acts on the particle with an uniform acceleration g in the $-z$ direction.

   (a) Find the Lagrangian for the system. (2 points)

   (b) Find the Euler-Lagrange equations. (2 points)

   (c) Assume the particle moves on a trajectory such that z is a constant, so that $z = h$. Find the energy and angular momentum in terms of m, h, a, and g. (2 points)

   (d) There is a small perturbation in the z direction. Find the frequency of oscillation about the unperturbed orbit in r, assuming very small oscillation amplitude. (4 points)

3. Consider a small bead of mass m that is constrained to lie on a rigid wire that is wound into a helix of radius R centered on the z-axis. The pitch of the helix is z_0, so that circling the z-axis exactly once counter-clockwise the bead would increase its vertical co-ordinate by z_0 . The bead is assumed to move upon the wire without friction but is subject to gravity, oriented in the z-directon (along the axis of the helix).

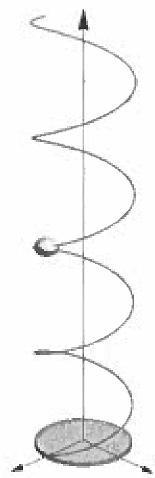


Figure 2: The bead slides without friction.

(a) Write the general form for the kinetic energy of a particle of mass m in cylindrical coordinates, (ρ, θ, z). (1 point)

(b) Construct a Lagrangian in terms of (ρ, θ, z) that includes the constraints required to confine the bead to move on the helix. Assuming the bead is released from rest, from a height h above the base of the helix, use this Lagrangian to calculate the motion of the bead as a function of time, and the generalized forces of constraint that act on the bead. For this part of the problem, the helix itself does not move. (3 points)

(c) Show that total mechanical energy of the bead is conserved. (1 point)

(d) Compute the time required for the bead to reach the base, and discuss how it depends upon the pitch. (2 points)

(e) The base of the helical wire is now affixed to a motor that generates a rotation of the wire about the z-axis at a fixed angular frequency ω_0. Recompute the time required for the bead to reach the base if it is again started from rest at a height h above the base. Discuss the effect of the rotation on the forces of constraint and the time it takes the bead to reach the base. (3 points)

Statistical Mechanics

4. A cylindrical container is initially separated by a clamped, thermally conductive piston into two compartments of equal volume. The left compartment is filled with one mole of neon gas at a pressure of four atmospheres and the right with argon gas at one atmosphere. The gases may be considered as ideal. The whole system is initially at temperature $T = 300\text{K}$, and is thermally insulated from the outside world. The heat capacity of the cylinder-piston system is C (a constant). The piston is now unclamped and released to move freely without friction. Eventually, due to slight dissipation, it comes to rest in an equilibrium position.

   (a) Find the new temperature of the system. (2 points)

   (b) Find the ratio of final neon to argon volumes. (2 points)

   (c) Find the total entropy change of the system. (2 points)

   (d) Find the additional entropy change which would be produced if the piston were removed. (2 points)

   (e) If, in the initial state, the gas in the left compartment were a mole of argon instead of a mole of neon, which, if any, of the answers to (A), (B) and (C) would be different? (2 points)

5. A statistical system is characterized by N distinguishable and non-interacting atoms in thermal equilibrium with a reservoir at temperature T. Each atom can occupy the energy levels $E_n = (n+1)\epsilon$, with $\epsilon > 0$ and $n = 0, 1, 2, ..., +\infty$. The degeneracy of the n-th level is equal to $g_n = \lambda^n$, with $\lambda > 1$.

   (a) Find the canonical partition function Z(T, N). (3 points)

   (b) Find the average energy U(T, N) for this system. (2 points)

   (c) Find the specific heat C(T, N) for this system. (2 points)

   (d) What happens to the specific heat at low temperatures? (1 point)

   (e) Is there a temperature above which the canonical description becomes invalid? If yes, what is this temperature, expressed as a function of λ and ϵ? (2 points)

6. An ideal paramagnet witch magnetic moments pointing in arbitrary directions is described by the Hamiltonian

$$\mathcal{H} = -\sum_{i=1}^{N} \vec{m}_i \cdot \vec{H} = -\sum_{i=1}^{N} mH \cos\theta_i$$

where the magnetic field $\vec{H}$ is non-zero only in the z-direction. The magnetic moments can be represented in terms of the spherical coordinates as:

$$\vec{m}_i(\theta_i, \phi_i) = m \left(\sin\theta_i \cos\phi_i \, \hat{\imath} + \sin\theta_i \sin\phi_i \, \hat{\jmath} + \cos\theta_i \, \hat{k} \right).$$

The phase-space of each moment consists of a unit sphere with volume element $d\Omega_i = \sin\theta_i \, d\theta_i \, d\phi_i$

(a) Using the canonical ensemble, calculate the Helmholtz free energy and the internal energy of the paramagnet. (3 points)

(b) Find the heat capacity

$$c_H = \left. \frac{\partial U}{\partial T} \right|_H .$$

Does it vanish as $T \to 0$? (2 points)

(c) Calculate the average magnetization $\langle \vec{m} \rangle$. (2 points)

(d) Calculate the magnetic susceptibility

$$\chi_{zz} = - \left. \frac{\partial^2 F}{\partial H_z^2} \right|_{T,N}$$

(2 points)

(e) The magnetic susceptibility is related to a measurable quantity of this model via the fluctuation-dissipation theorem. To what quantity is it proportional and why? (1 point)

Problem 1

(a) Moment of inertia of the rod:

With linear mass density $\frac{M}{L}$, a segment of length dx has a mass $dm = \frac{M}{L} dx$. Then

$$I_r = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \, dm = \frac{M}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \, dx$$

$$= \frac{M}{L} \frac{1}{3} x^3 \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{1}{12} \frac{M L^3}{L} = \frac{1}{12} M L^2$$

(b) The angular acceleration is determined by

$$\tau = I_0 \alpha \Rightarrow \alpha = \frac{\tau}{I_0}$$

where I_0 is the total moment of inertia.

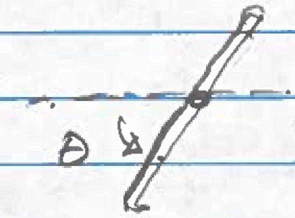
$$I_0 = \frac{1}{12} M L^2 + m \left(\frac{L}{4}\right)^2$$

$$= \left(\frac{M}{12} + \frac{m}{16}\right) L^2 = \frac{1}{4} \left(\frac{M}{3} + \frac{m}{4}\right) L^2$$

The initial torque is $mg\frac{L}{4}$ so that

$$\alpha = \frac{\frac{1}{4}mgL}{\frac{1}{4}\left(\frac{M}{3}+\frac{m}{4}\right)L^2} = \frac{12m}{(4M+3m)}\frac{g}{L}$$

(c)



If the rod rotates by an angle θ then the bead drops a vertical distance $\frac{L}{4}\sin\theta$. This is converted into kinetic energy

$$E_i = E_f.$$

$$0 = \frac{1}{2}I_o\omega_f^2 - mg\frac{L}{4}\sin\theta$$

So

$$\omega_f^2 = \frac{2}{I_o}\frac{mgL}{4}\sin\theta$$

$$\omega_f = \sqrt{\frac{mgL}{2I_o}}\sqrt{\sin\theta}$$

At $\theta = \frac{\pi}{2}$ we have $\sin\theta = 1$

$$\omega_f = \sqrt{\frac{mgL}{2I_0}}$$

(d) The total linear acceleration $|\vec{a}|$ is

given by

$$a^2 = a_r^2 + a_\theta^2$$

$$a_r = -\omega^2\left(\frac{L}{4}\right) = -\frac{mgL}{8I_0}\sin\theta$$

The acceleration in the θ direction is

$$a = \frac{L}{4}\alpha$$

We need α as a function of θ. From (c)

$$\alpha = \frac{d}{dt}\omega = \frac{d}{dt}\sqrt{\frac{mgL}{2I_0}}\sqrt{\sin\theta}$$

$$= \sqrt{\frac{mgL}{2I_0}}\,\frac{1}{2}\,\frac{\cos\theta}{\sqrt{\sin\theta}}\,\frac{d\theta}{dt}$$

But $\frac{d\theta}{dt} = \omega$, which we know:

$$\alpha = \sqrt{\frac{mgL}{2I_0}} \cdot \frac{1}{2} \cdot \frac{\cos\theta}{\sqrt{\sin\theta}} \sqrt{\frac{mgL}{2I_0}} \sqrt{\sin\theta}$$

$$= \frac{1}{2} \cdot \frac{mgL}{2I_0} \cos\theta = \frac{1}{4} \frac{mgL}{I_0} \cos\theta$$

So

$$a_\theta = \frac{L}{4}\alpha = \frac{mgL}{16} \cos\theta$$

And

$$a^2 = \left(\frac{mgL}{I_0}\right)^2 \left[\frac{\sin^2\theta}{64} + \frac{\cos^2\theta}{256}\right]$$

$$= \frac{1}{64}\left(\frac{mgL}{I_0}\right)\left[\sin^2\theta + \frac{\cos^2\theta}{4}\right]$$

a is a maximum when a^2 is a maximum - so

$$\frac{d}{d\theta} a^2 = 2\left[\sin\theta\cos\theta - \frac{\cos\theta\sin\theta}{4}\right] = 0$$

We get answers $\theta = 0$ & $\theta = \frac{\pi}{2}$.

Direct evaluation shows $\frac{\pi}{2}$ is the max.

(e) Since the bead slides straight down gravity does not exert a torque. The total angular momentum is conserved. Since the bead slides out, we know the new moment of inertia, I_1 is larger than I_0.

$$L_i = L_f$$

$$\omega_i I_0 = \omega_f I_1$$

$$\omega_f = \omega_i \frac{I_0}{I_1} < \omega_i$$

because $\frac{I_0}{I_1} < 1$.

(f) The final kinetic energy is

$$K_f \quad \frac{1}{2} I_1 \omega_f^2 = \frac{1}{2} I_1 \left(\omega_i \frac{I_0}{I_1} \right)^2$$

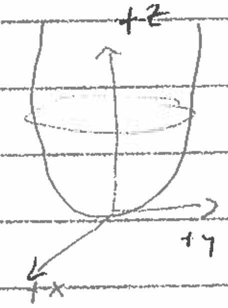
$$= \frac{1}{2} I_0 \omega_i^2 \cdot \left(\frac{I_0}{I_1} \right)$$

$$= K_i \left(\frac{I_0}{I_1} \right) < K_i$$

$\boxed{\text{Or}}$

$$K_f = \frac{1}{2} \frac{L^2}{I_1} \qquad K_i = \frac{1}{2} \frac{L^2}{I_0}$$

But $I_1 > I_0$ so $K_f < K_i$. Where did the energy go? The gravitational potential energy did work to slow down the rotation!

(a) 

$$z = \tfrac{1}{2} a r^2$$

$$T = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\theta}^2 + \tfrac{1}{2} m \dot{z}^2$$

$$V = m g z$$

$$\dot{z} = a r \dot{r}$$

$$\mathcal{L} = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} m r^2 \dot{\theta}^2 + \tfrac{1}{2} m a^2 r^2 \dot{r}^2 - \tfrac{1}{2} m g a r^2$$

(b) $$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m \dot{r} + m a^2 r^2 \dot{r} \qquad \frac{d}{dt} \rightarrow m \ddot{r} + m a^2 r^2 \ddot{r} + 2 m a^2 r \dot{r}^2$$

$$\frac{\partial \mathcal{L}}{\partial r} = m r \dot{\theta}^2 + m a^2 r \dot{r}^2 - m g a r$$

$$\cancel{m} \ddot{r} + \cancel{m} a^2 r^2 \ddot{r} + \cancel{2} \cancel{m} a^2 r \dot{r}^2 - \cancel{m} r \dot{\theta}^2 - \cancel{m a^2 r \dot{r}^2} + \cancel{m} g a r = 0$$

$$(1 + a^2 r^2) \ddot{r} - a^2 r \dot{r}^2 - r \dot{\theta}^2 + g a r = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} \qquad \frac{d}{dt} \rightarrow m r^2 \ddot{\theta}, \qquad \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$m r^2 \ddot{\theta} = 0 \qquad L = m r^2 \dot{\theta} = \text{constant}$$

(c) Z is fixed to height h, What is energy and angular momentum in terms of h, ~~a~~, and g.

$r = r_0$, $\dot{r} = \ddot{r} = 0$ $\quad h = \frac{1}{2} a r_0^2$ $\quad \dot{z} = 0$, $z = h$

Lagrangian becomes $-r_0\dot{\theta}^2 + g a r_0 = 0$

$$\dot{\theta}^2 = ga$$

$$\text{Energy} = T + V = \frac{1}{2} m r_0^2 \dot{\theta}^2 + mgh$$

$$= \frac{1}{2} m r_0^2 g a + mgh$$

$$= \frac{1}{\cancel{2}} m g \cancel{a} \left(\frac{2h}{\cancel{a}}\right) + mgh$$

$r_0^2 = \frac{2h}{a}$ $\qquad = mgh + mgh \quad = \underline{2mgh}$

Angular momentum : $L = mr^2\dot{\theta} = m\left(\frac{2h}{a}\right)\sqrt{ga}$

$$\underline{L = 2mh\sqrt{\frac{g}{a}}}$$

(d) Let $r = r_0 + \epsilon$ where $\epsilon << r$

$\dot{r} = \dot{\epsilon}$, $\ddot{r} = \ddot{\epsilon}$, $\dot{\epsilon}^2 = 0$

$r^2 = r_0^2 + r_0\epsilon + \epsilon^2 \simeq r_0^2$

Now,

$$\mathcal{L} = (1+a^2 r_0^2)\ddot{\epsilon} - r\dot{\theta}^2 + gar = 0$$

Angular momentum is conserved because to torque around z. From before, $L^2 = m^2 r_0^4 \dot{\theta}^2 = m^2 r_0^4 ga$

Now, $r\dot{\theta}^2 = \dfrac{m^2 r^4 \dot{\theta}^2}{m^2 r^3} = \dfrac{L^2}{m^2 r^3} = \dfrac{\cancel{m^2} r_0^4 ga}{\cancel{m^2} r^3}$

$= \dfrac{r_0^4 ga}{(r_0+\epsilon)^3} = \dfrac{r_0 ga}{(1+\epsilon/r_0)^3} = r_0 ga(1+\epsilon/r_0)^{-3} \simeq r_0 ga - 3ga\epsilon$

$$(1+a^2r_0^2)\ddot{\epsilon} - \cancel{r_0 ga} + 3ga\epsilon + \cancel{gar_0} + ga\epsilon = 0$$

$$(1+a^2r_0^2)\ddot{\epsilon} + 4ga\epsilon = 0 \qquad r_0^2 = \frac{2h}{a}$$

$$a^2 r_0^2 = 2ah$$

$$\ddot{\epsilon} = -\frac{4ga}{(1+2ah)}\epsilon$$

$$\omega = \sqrt{\frac{4ga}{1+2ah}}$$

a) The bead's motion is described by 3 generalized co-ordinates: r, φ, z

There exist 2 constraints imposed by the spiral wire:

$$f_1 = r - R = 0$$

$$f_2 = \varphi - \varphi_0 - kz = 0$$

This defines the orientation of the wire. In part (d) it will become $\varphi_0 = \omega t$

The Lagrangian is:

$$T = \tfrac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2 \right)$$

$$V = mgz$$

$$L = T - V.$$

The forces of constraint are obtained using the method of Lagrange multipliers,

EOM [from L=T-V]

$r \Rightarrow \qquad m\ddot{r} - mr\dot{\varphi}^2 = \sum_j \lambda_j \frac{\partial f_j}{\partial r}$

$$\Rightarrow \quad \boxed{m\ddot{r} - mr^2\dot{\varphi}^2 = \lambda_1} \qquad (1)$$

$\varphi \Rightarrow \qquad \frac{d}{dt}\left(mr^2\dot{\varphi}\right) = \sum_j \lambda_j \frac{\partial f_j}{\partial \varphi} = \lambda_2$ [see (b)]

or equivalently,

$$\boxed{2mr\dot{r}\dot{\varphi} + mr^2\ddot{\varphi} = \lambda_2} \qquad (2)$$

$$z \Rightarrow \quad \boxed{m\ddot{z} + mg = -\lambda_2} \qquad (3)$$

So we have 3EOM + 2 constraint eqns.

$\Rightarrow$ From f_1 : $\boxed{\dot{r} = \ddot{r} = 0}$ (4)

From f_2 :

$$\boxed{\begin{aligned} \dot{\varphi} &= kz + \cancel{\dot{\varphi}_0} \rightarrow 0 \text{ for (a)} \\ \ddot{\varphi} &= k\ddot{z} + \cancel{\ddot{\varphi}_0} \rightarrow 0 \end{aligned}} \qquad (5)$$

Plugging (5) → (3):

$$m\frac{\ddot{\varphi}}{k} + mg = -k\lambda_2 \quad \Rightarrow \quad \lambda_2 = -\frac{m}{k^2}\ddot{\varphi} - \frac{m}{k}g$$

This result can be equated to (2) using that $\dot{r} = \ddot{r} = 0$,

$$m r^2 \ddot{\varphi} = -\frac{m}{k^2}\ddot{\varphi} - \frac{m}{k}g$$

$$\Downarrow \text{ rearrange}$$

$$\ddot{\varphi} = -\gamma \quad \text{w/} \quad \gamma = \frac{gk}{k^2R^2+1}$$

We can formally integrate the eqn for $\ddot{\varphi}$,

$$\dot{\varphi} = -\gamma t + a \qquad [a = \dot{\varphi}(0)]$$

$$\varphi = -\frac{\gamma}{2}t^2 + at + b \qquad [b = \varphi_0 + k z(0)]$$

With the full solution of $\varphi(t)$ we can obtain the relevant Lagrange multipliers:

$$(1) \longrightarrow \quad \lambda_1 = -mR(a-\gamma t)^2 \qquad \left(\begin{array}{c}\dot{r}=0 \\ \& \ r=R\end{array}\right)$$

$$(2) \longrightarrow \quad \lambda_2 = -mR^2\gamma$$

The generalized forces of constraint are:

$$F_r = -\lambda_1 \frac{\partial f_1}{\partial r} = mR(a-\gamma t)^2$$

$$F_\varphi = -\lambda_2 \frac{\partial f_2}{\partial \varphi} = mR^2\gamma \quad \propto g$$

$$F_z = -\lambda_2 \frac{\partial f_2}{\partial z} = -mR^2 k \gamma \quad \propto g$$

F_r supplies the radial force associated w/ the wire and grows w/ time as the bead accelerates w/ gravity.

F_φ & F_z are proportional due to gravity as they relate to the wires response opposing gravity.

b) One can show that energy is conserved by: ~~compu~~

i) $\frac{\partial L}{\partial t} = 0$

ii) computing the energy function $h(\vec{q}, \dot{\vec{q}}, t)$ & showing $h = T+V$.

i) implies $h = T+V$ is conserved.

Explicitly using our constraints, the Lagrangian in a) becomes:

$$L = \frac{m}{2}\left(R^2 k^2 \dot{z}^2 + \dot{z}^2 \right) - mgz \Rightarrow \boxed{\frac{\partial L}{\partial t} = 0}$$

Then:

$$h = \dot{z}\frac{\partial L}{\partial \dot{z}} - L$$

$$= m\left(k^2R^2+1\right)\dot{z}^2 + mgz - \frac{m}{2}\left(k^2R^2+1\right)\dot{z}^2$$

$$= T+V$$

$\therefore$ energy is conserved.

c)

To work out the fall time, use that:

$$z = \varphi / k \qquad (\text{set } \varphi_0 \to 0)$$

$$= -\frac{\gamma}{2k} t^2 + \frac{a}{k} t + b/k$$ encodes $z(0)$.

Solving for t:

$$t = \frac{-a/k \pm \sqrt{\frac{a^2}{k^2} + \frac{4b\gamma}{2k^2}}}{-\gamma/k}$$

The bead is initially at rest $\Rightarrow$ $a = 0$

$$\therefore \quad t = \sqrt{\frac{2b}{\gamma}} \qquad (\text{take +ve solution})$$

From a): $b = \varphi_0 + k z(0) = kh \qquad (\varphi_0 = 0)$

$$\therefore \quad t = \sqrt{\frac{2kh}{\gamma}}$$

t : increases w/ k. increasing $\Rightarrow$ relate to g, R
decreases w/ γ increasing etc.

d) The solution follows similarly to a) but now $\varphi_0 \to \omega t$, to account for the ~~t~~ time-dependent orientation.

Revisiting a) we importantly have:

$$⑤ \Rightarrow ⑤' \quad \boxed{\begin{aligned} \dot{\varphi} &= k\dot{z} + \omega \\ \ddot{\varphi} &= k\ddot{z} \end{aligned}} \leftarrow \dot{\varphi}_0 = \omega$$

$⑤' \Rightarrow ③$ still yields,

$$\ddot{\varphi} = -\gamma \qquad w/ \qquad \gamma = \frac{gk}{k^2R^2+1}$$

but now, solution of the differential equation gives:

$$\varphi = -\frac{\gamma}{2}t^2 + \omega t + kz(0) \qquad \left[\begin{aligned} \dot{\varphi}(0) &= \omega \\ \varphi(0) &= kz(0) \end{aligned}\right]$$

From $\varphi(t)$ we can obtain our Lagrange multipliers,

$$① \to \lambda_1 = -mR(\omega - \gamma t)^2$$

$$② \to \lambda_2 = -mR^2\gamma$$

So that in terms of the generalized forces of constraint,

$$F_\varphi + F_z \Rightarrow \text{unchanged by rotation.}$$

$$F_r \Rightarrow mR(\omega - \gamma t)^2$$

rotation enters through here.

To get the fall time, we use ⑤$'$ to obtain

$$\ddot{z} = \frac{\ddot{\varphi}}{k} = -\frac{\gamma}{k}$$

which can be integrated to obtain

$$z = z(0) - \frac{\gamma}{2k}t^2$$

Hence, the rotation does <u>not</u> enter into $z(t)$, and thus the fall time remains identically,

$$t = \sqrt{\frac{2kh}{g}} \quad !$$

Solution

A 2 pts) The new temperature of the system (the piston is thermally conductive).
The internal energy of an ideal gas is a function only dependent on temperature, so the internal energy of the total system does not change. Therefore neither does the temperature. The new equilibrium temperature is $T = 300$ K.

B 2 pts) The ratio of final neon to argon volumes.
The volume ratio is the ratio of molecular numbers, and is also the ratio of initial pressures.

$$\frac{P_i^{Ar}V_i^{Ar}}{n^{Ar}} = \frac{P_i^{Ne}V_i^{Ne}}{n^{Ne}} = RT = \frac{P_f^{Ar}V_f^{Ar}}{n^{Ar}} = \frac{P_f^{Ne}V_f^{Ne}}{n^{Ne}} \Rightarrow \frac{V_f^{Ar}}{n^{Ar}} = \frac{V_f^{Ne}}{n^{Ne}} \Rightarrow \frac{V_f^{Ne}}{V_f^{Ar}} = \frac{n^{Ne}}{n^{Ar}} = \frac{4}{1} \Rightarrow n^{Ar} = \frac{1}{4}$$

C 2pts) The total entropy change of the system.

$$\Delta S = n_{Ne}R\ln\left(\frac{V_f^{Ne}}{V_i^{Ne}}\right) + n_{Ar}R\ln\left(\frac{V_f^{Ar}}{V_i^{Ar}}\right)$$

$$= R\ln\left(\frac{\frac{4}{5}}{\frac{1}{2}}\right) + \frac{1}{4}R\ln\left(\frac{\frac{1}{5}}{\frac{1}{2}}\right) = R\ln\left(\frac{8}{5}\right) + \frac{1}{4}R\ln\left(\frac{2}{5}\right) = 2.0 \text{ J/K}$$

D 2pts) The additional entropy change which would be produced if the piston were removed.

$$\Delta S_{mix} = n_{Ne}R\ln\left(\frac{V_f^{Ne}}{V_i^{Ne}}\right) + n_{Ar}R\ln\left(\frac{V_f^{Ar}}{V_i^{Ar}}\right)$$

$$= R\ln\left(\frac{1}{\frac{4}{5}}\right) + \frac{1}{4}R\ln\left(\frac{1}{\frac{1}{5}}\right) = R\ln\left(\frac{5}{4}\right) + \frac{1}{4}R\ln(5) = 5.2 \text{ J/K}$$

E 2pts) If, in the initial state, the gas in the left compartment were a mole of argon instead of a mole of neon, which, if any, of the answers to (A), (B) and (C) would be different?

Only C will change because the entropy of mixing will be zero.

a) distinguishable particles, so

$$Z(T,N) = (Z_1(T))^N$$

$\uparrow$ single atom partition fctn.

Here, $Z_1(T) = \sum_{n=0}^{\infty} \lambda^n e^{-\beta(n+1)\varepsilon}$ $\quad \left(\beta = \frac{1}{kT},\ k = \text{Boltzmann const.}\right)$

$$= e^{-\beta\varepsilon} \sum_{n=0}^{\infty} e^{-(\beta\varepsilon - \ln\lambda)n} \quad \text{/ geometric sum}$$

$$= \frac{e^{-\beta\varepsilon}}{1 - e^{-(\beta\varepsilon - \ln\lambda)}}$$

$$= \frac{e^{-\beta\varepsilon}}{1 - e^{-\beta\varepsilon}\lambda}$$

$$= \frac{1}{e^{\beta\varepsilon} - \lambda}$$

$$\Rightarrow Z(T,N) = (Z_1(T))^N = \left(\frac{1}{e^{\beta\varepsilon} - \lambda}\right)^N$$

b) $$u = -\left(\frac{\partial \ln Z}{\partial \beta}\right)_N$$

$$= -\left(\frac{\partial(-N\ln(e^{\beta\varepsilon} - \lambda))}{\partial \beta}\right)_N$$

$$= N\frac{1}{e^{\beta\varepsilon} - \lambda} \cdot (\varepsilon e^{\beta\varepsilon}) = \frac{N\varepsilon}{1 - \lambda e^{-\beta\varepsilon}}$$

c) $C(T,N) = \left(\frac{\partial U}{\partial T}\right)_N$

$$= \left[\frac{\partial}{\partial T}\left(\frac{N\varepsilon}{1-\lambda e^{-\beta\varepsilon}}\right)\right]_N$$

$$= -\frac{N\varepsilon}{(1-\lambda e^{-\beta\varepsilon})^2}\cdot\left(\varepsilon\lambda e^{-\beta\varepsilon}\right)\cdot\underbrace{\frac{\partial\beta}{\partial T}}_{-\frac{1}{kT^2}}$$

$$= +\frac{N\varepsilon^2\lambda e^{-\beta\varepsilon}}{kT^2(1-\lambda e^{-\beta\varepsilon})^2}$$

d) $\lim_{T\to 0} C = 0$ because $e^{-\beta\varepsilon} \to 1$ and $\frac{1}{kT^2} \to 0$

c) There is such a temperature T_c, because the geometric sum in Z diverges if $e^{-\beta\varepsilon - \ln\lambda} \geq 1$

This happens when $\ln\lambda \geq \beta\varepsilon$ so

$T \geq \frac{\varepsilon}{k\ln\lambda} (= T_c)$ causes the description to be invalid.

Integrate to get

$$K\ln\frac{T}{T_0} = \alpha(\frac{(2L_0)^2}{L_0} + \frac{L_0^2}{2L_0} - \frac{3}{2}L_0) = \alpha L_0. \tag{15}$$

Hence $T_f = T_0 e^{\alpha L_0/K}$.

g) Previously we have known that $\frac{\partial S}{\partial x}|_T = -\alpha(\frac{x}{L_0} - \frac{L_0^2}{x^2})$. Since $\alpha > 0$, $\frac{\partial S}{\partial x}|_T < 0$, and when stretched at a constant T, the entropy of the rubber band decreases.

Problem 2:

a) The partition function is

$$Z = \prod_{i=1}^{N} \int d\Omega_i e^{\beta m H \cos\theta_i} = (2\pi \int_{-1}^{1} \sin\theta_i d\theta e^{\beta m H \cos\theta_i})^N = (\frac{2\pi}{\beta m H}\sinh(\beta m H))^N = Z_m^N. \tag{16}$$

The Helmholtz free energy is

$$F = -k_B T \log Z = N k_B T \log(\beta m H) N k_B T \log(4\pi \sinh(\beta m H)). \tag{17}$$

The internal energy is

$$U = -\frac{\partial}{\partial\beta}\log Z = N k_B T - N m H \coth(\beta m H). \tag{18}$$

b) From a), the heat capacity is

$$c_H = \frac{\partial U}{\partial T}\Big|_H = N k_B - N k_B (\frac{mH}{k_B T \sinh(mH/k_B T)})^2. \tag{19}$$

When $T \to 0$, $c_H \to N k_B$, and is non-vanishing. It corresponds to the contribution of the potential energy two-dimensional harmonic oscillator (the fluctuating magnetic moments nearly aligned with the magnetic field feel a harmonic potential).

c) Since $Z = Z_m^N$, the magnetization can be written as

$$\langle \mathbf{m} \rangle = \frac{1}{Z_m}\int d\Omega \mathbf{m}(\theta,\phi) e^{\beta m H \cos\theta} \tag{20}$$

. So $\langle m_x \rangle = \langle m_y \rangle = 0$, and

$$\langle m_z \rangle = \frac{2\pi m}{Z_m}\int_{-1}^{1} \sin\theta d\theta \cos\theta e^{\beta m H \cos\theta} = m\coth(\frac{mH}{k_B T}) - \frac{k_B T}{H}. \tag{21}$$

Using the expression of F in a), it is straightforward to show $\langle \mathbf{m} \rangle = -\frac{1}{N}\frac{\partial F}{\partial \mathbf{H}}|_{T,N}$.

d) We can find χ_{zz} by deriving $\langle m_z \rangle$ with respect to H. This leads to

$$\chi_{zz} = \frac{N m^2}{k_B T}\left((\frac{k_B T}{mH})^2 - \frac{1}{\sinh^2(mH/k_B T)}\right). \tag{22}$$

e) Similar to c), we find

$$\langle m_z^2 \rangle = \frac{2\pi m}{Z_m}\int_{-1}^{1} \sin\theta d\theta \cos^2\theta e^{\beta m H \cos\theta} = -\frac{2k_B T}{H}(m\coth(\frac{mH}{k_B T}) - \frac{k_B T}{H}) + m^2. \tag{23}$$

Using the results in c) and d), it is straightforward to show

$$(\Delta m_z)^2 = \langle m_z^2 \rangle - \langle m_z \rangle^2 = \frac{k_B T}{N}\chi_{zz}. \tag{24}$$