E & M Qualifier

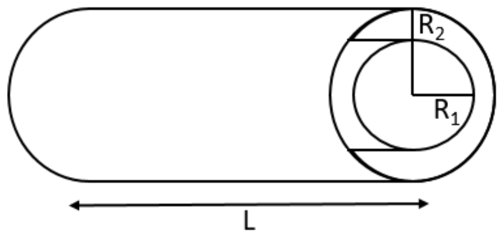
January 2021

To insure that your work is given appropriate credit you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1

Two concentric conducting cylinders are arranged as shown in the diagram, with R_1 the radius of the inner cylinder, R_2 the radius of the outer cylinder, and L the length of the cylinders. We'll assume $R_1 < R_2 \ll L$.



(a) We place a charge of $-Q$ on the inner cylinder and $+Q$ on the outer cylinder. Explicitly use Gauss's Law to determine the electric field in the three regions ($r < R_1$, $R_1 < r < R_2$, and $R_2 < r$). Assume the charge is uniformly distributed along the cylinders. [2 points]

(b) Use the electric field you just calculated to determine the potential difference between the cylinders. [3 points]

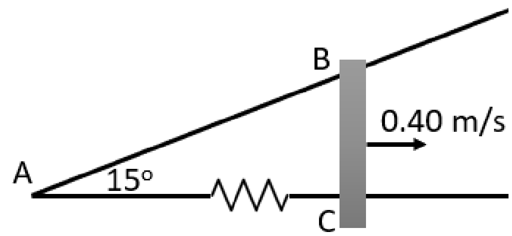
(c) Use the potential difference you just calculated to determine the amount of work required to move an infinitesimal amount of charge dq from the inner cylinder to the outer cylinder. [1 point]

(d) We can think of the energy stored on this capacitor as the amount of work required to move a total charge Q from the inner cylinder to the outer cylinder. Use the work required to move a charge dq to determine the total energy stored on this capacitor. [2 points]

(e) Without doing any complex calculations, briefly discuss how your answers to these questions would change if we inserted a dielectric with permittivity ϵ between the cylinders (e.g. would they increase, decrease, or stay the same?). Include an explanation for why those changes would happen. [2 points]

Problem 2

A copper rod is sliding on two conducting rails that make an angle of 15° with respect to each other, as in the drawing. The rod is moving to the right with a constant speed of 0.40 m/s. A 0.42T uniform magnetic field is perpendicular to the plane of the paper pointing out of the page. A small resistor is included in one of the rails. This system is shown in the diagram below.



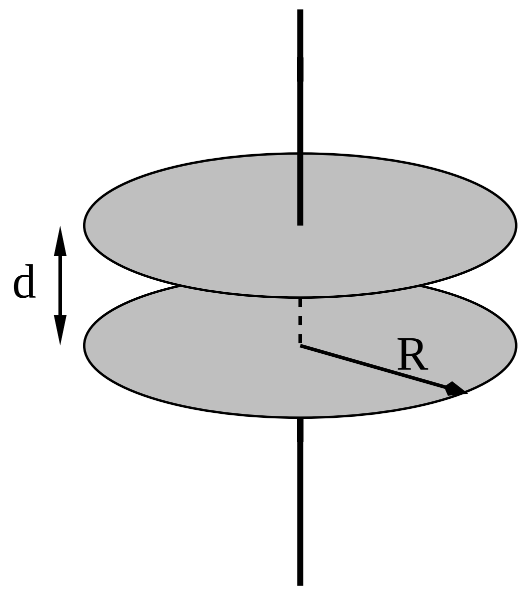
(a) Determine the magnitude of the average emf induced in the triangle ABC during the 5.0s period after the rod has passed point A. [4 points]

(b) If the resistor has a resistance of 5.0 Ohms, what is the current passing through the resistor and which way does that current flow? Assume the resistance of the rails and the rod are all negligible. [3 points]

(c) As current flows through the resistor, it dissipates energy. Where does that energy come from during this process? Briefly describe the flow of energy through this system. [3 points]

Problem 3

Consider a parallel-plate capacitor with circular plates of radius R. A voltage of $V(t) = V_o \sin \omega t$ is applied across the capacitor by connecting a wire to each of the plates. We will use cylindrical coordinates with the z-axis passing through the center of both capacitor plates. The magnetic field between the plates is measured to be in the azimuthal direction. Its magnitude is found to be proportional to s (the distance from the z-axis) and proportional to $\cos \omega t$. The distance between the plates is d and the material between the plates is a non-magnetic linear dielectric with a permittivity of $3\epsilon_o$.



(a) Find the magnitude and direction of the electric field between the plates. [2 points]

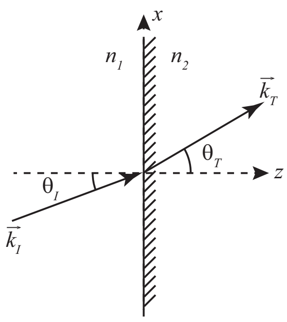
(b) Use the Ampere-Maxwell equation to find the proportionality constant that relates B to $s \cos \omega t$. [3 points]

(c) Use the Ampere-Maxwell equation and your answer to part b to find the current through the wires that are attached to the capacitor. [3 points]

(d) Find the magnitude and direction of the Poynting vector for the capacitor. Then find the power that flows into the capacitor. [2 points]

Problem 4

Consider an interface between a high index of refraction (n_1) medium and a low index of refraction medium (n_2) such that $n_1 > n_2$. A plane wave with frequency ω is incident from the high index medium on such an interface at an angle θ_I, as shown in the figure below.



(a) Use Snell's law to show that when $\theta_I = \theta_c \equiv \sin^{-1}(n_2/n_1)$ the angle of the transmitted wave is $\theta_T = 90°$. [1 point]

(b) Consider now the case in which $\theta_I > \theta_c$. Show that for this condition the transmitted field takes the form

$$\vec{E}_T(\vec{r}, t) = \vec{E}_{0T} e^{-\kappa z} e^{i(kx - \omega t)}, \tag{1}$$

where

$$\kappa = \frac{\omega}{c}\sqrt{(n_1 \sin\theta_I)^2 - n_2^2} \qquad \text{and} \qquad k = \frac{\omega n_1}{c}\sin\theta_I,$$

with c representing the speed of light. [Hint: use Snell's law even though θ_T can no longer be interpreted as an angle in this case.] [3 points]

(c) What is the physical meaning of the form of the transmitted field in part (b)? [1 point]

(d) For an electric field polarized perpendicular to the plane of incidence (s-polarized light), show that when $\theta_I > \theta_c$ the real transmitted fields take the form

$$\begin{aligned} \vec{E}_T(\vec{r}, t) &= E_{0T} e^{-\kappa z} \cos(kx - \omega t)\hat{y}, \\ \vec{B}_T(\vec{r}, t) &= \frac{E_{0T}}{\omega} e^{-\kappa z}[\kappa \sin(kx - \omega t)\hat{x} + k\cos(kx - \omega t)\hat{z}], \end{aligned}$$

where the amplitude E_{0T} is taken to be real and the hat indicates a unit vector. [2 points]

(e) For the fields in part (d), calculate the Poynting vector and show that on average no energy is transmitted in the z direction. [3 points]

Problem 5

You are in a spaceship and fly by your friends standing on an asteroid. Your friends are using some clever technology to generate a homogenous, time-independent, purely magnetic field $B_x = 0.1$ T, $B_y = 0.2$ T, $B_z = 0.5$ T in their rest-frame K and there is no electric field in their rest-frame. Consider your own rest-frame K', from which you see the asteroid with your friends fly by at a velocity of $\mathbf{v}_K = -0.75\, c\, \hat{\mathbf{k}}$. Here, $\hat{\mathbf{k}}$ is a unit vector pointing in the z-direction and c the speed of light in vacuum. For this problem, use the following metric: $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$, and $g_{ij} = 0$ for $i \neq j$.

Hints: Use the following fields in terms of potentials and 4-vector potential

$$\begin{aligned}
\mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \\
\mathbf{B} &= \nabla \times \mathbf{A} \\
A^\mu &= (V/c, A_x, A_y, A_z)
\end{aligned}$$

(a) Explicitly derive the field strength tensor $F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$ in frame K in matrix form from the 4-vector potential A^μ. Hint: Make use of the fact that in frame K, there is no electric field and the magnetic field is time-independent. [2 points]

(b) Find the matrix form of a Lorentz boost $L^\alpha{}_\beta$ in the positive z-direction that would bring you into the frame K' from which frame K is seen to move in the negative z-direction with velocity $\mathbf{v}_K = -0.75\, c\, \hat{\mathbf{k}}$. At time zero, the origins of both frames overlap, and the clocks are synchronized then. [2 points]

(c) Explicitly verify that the boost $L^\alpha{}_\beta$ leaves the metric g invariant. HINT: evaluate the matrix product $L^T\, g\, L$ and show that it is equal to g. [2 points]

(d) Apply the boost $L^\alpha{}_\beta$ to $F^{\mu\nu}$ to find $F'^{\mu\nu}$, the field strength tensor as seen in the moving frame K'. [2 points]

(e) What are the magnetic field components B'_x, B'_y, and B'_z, seen in frame K', as functions of the magnetic field components B_x, B_y, and B_z, seen in frame K? What are the numerical values you would measure in your spaceship, in Tesla? [1 point]

(f) What are the electric field components E'_x, E'_y, and E'_z seen in frame K', as functions of the magnetic field components B_x, B_y, and B_z, seen in frame K? What are the numerical field values you would measure in your spaceship, in Volts per meter? [1 point]

Problem 6

(a) Write down the four Maxwell equations in three-vector notation. [2 points]

(b) Write $\vec{E}$ and $\vec{B}$ in terms of the potentials Φ and $\vec{A}$. What advantage is gained by use of the potentials? [2 points]

(c) Re-write Gauss' law and Ampere's law in terms of the potentials. *Hint*: $\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \vec{\nabla}^2 \vec{a}$. [2 points]

(d) Identify the Lorenz gauge condition and use it to decouple these equations. [1 point]

(e) Write down the Green function for the wave equation $\vec{\nabla}^2 G(\vec{x}, t; \vec{x}', t') - \frac{1}{c^2} \frac{\partial^2 G(\vec{x}, t; \vec{x}', t')}{\partial t^2} = -4\pi \delta^3(\vec{x} - \vec{x}')\delta(t - t')$ [1 point]

(f) Use the retarded Green function to write down the solutions Φ and $\vec{A}$ as integral equations. [1 point]

(g) Explain how these solutions exhibit causality in electrodynamics. [1 point]