E & M Qualifier

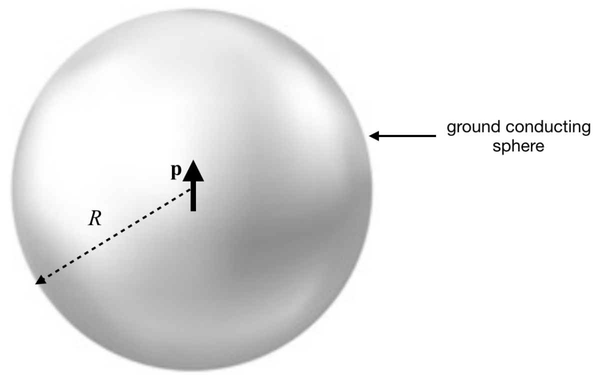
January, 2022

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1: Electrostatics

A pure dipole (dipole moment $\vec{p}$) pointing in the z-direction is located at the center of a grounded conducting sphere of radius R, as shown in the figure.



(a) What is the potential due to the dipole in the absence of the conducting sphere? Write your answer in terms of Legendre polynomials. [2 points]

(b) Write down the general solution to Laplace's equation in spherical coordinates and the boundary conditions that you need to find the potential inside and outside the sphere. [2 point]

(c) Use the boundary conditions and the fact that the potential close to origin should look like the one of the dipole to find the potential inside and outside the sphere. [4 points]

(d) Calculate the induced surface charge density on the conducting sphere and sketch the charge distribution on the surface of the sphere. Explain why one would expect such a charge distribution. [2 points]

Hint: Legendre polynomials

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= (3x^2 - 1)/2 \\
P_3(x) &= (5x^3 - 3x)/2 \\
P_4(x) &= (35x^4 - 30x^2 + 3)/8 \\
P_5(x) &= (63x^5 - 70x^3 + 15x)/8
\end{aligned}
$$

Problem 2: Magnetostatics

A long cylinder of radius a and permeability μ is placed in a uniform magnetic field $\vec{B}_0$ along the x-axis such that the cylinder is at right angles to $\vec{B}_0$ as shown in the figure below. The cylinder is along the z-axis.

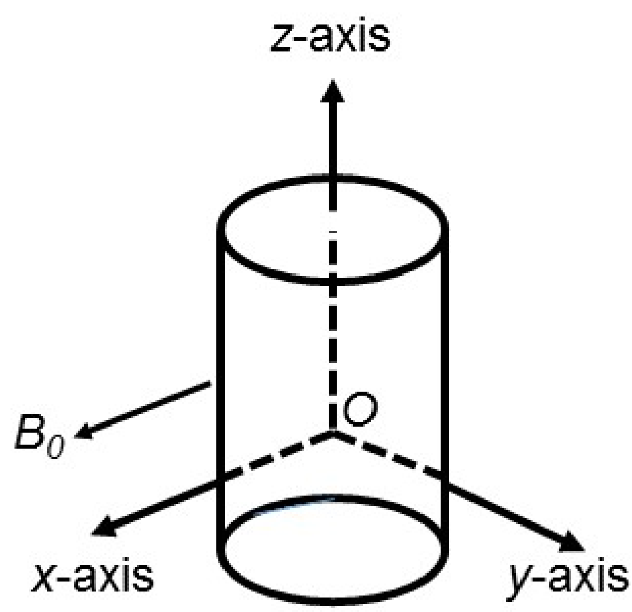
(a) Find the magnetic scalar potential U, defined as $\vec{H} = -\vec{\nabla}U$, *outside* (region 1, $\rho > a$) the cylinder.[3 points]

(b) Find the magnetic scalar potential U, defined as $\vec{H} = -\vec{\nabla}U$, *inside* (region 2, $\rho < a$) the cylinder. [3 points]

Hint: Expand U in terms of the $\cos\phi$ cylindrical harmonics: $\{\rho^n \cos n\phi, \rho^{-n} \cos n\phi\}$

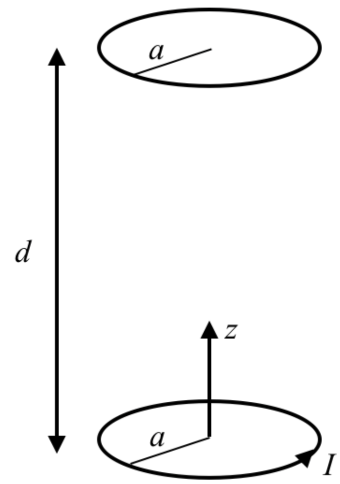
(c) Find the magnetic field $\vec{B}_2$ inside the cylinder. Sketch the lines of $\vec{B}_2$. [2 points]

(d) Based on the result you found in part (c) calculate the induced magnetization $\vec{M}$ inside the cylinder. Interpret your result in the cases where $\mu > \mu_0$ and $\mu < \mu_0$. [2 points]



Problem 3: Faraday's Law

A copper ring of radius a is at a fixed distance d (with $a << d$) directly above an identical copper ring. Each ring has a resistance R for circulating currents. An increasing current $I = I_0 t/\tau$ (in the counterclockwise direction when viewed from above) is applied to the lower ring, where I_0 and τ are constants. Neglect the self-inductance of each ring and make appropriate approximations.



(a) Use the Biot-Savart law to find the magnetic field along the axis of the lower ring. The field should be proportional to $(z^2 + a^2)^{-3/2}$ where the coordinate origin is at the center of the lower ring. [3 points]

(b) Using the result from part (a) to approximate the magnetic field at all points within the upper ring, find the induced emf in the upper ring. [2 points]

(c) Find the magnetic moment of the upper ring. [2 point]

(d) Find the mutual inductance between the two rings. [2 points]

(e) Do the rings attract or repel each other? Explain your answer. [1 point]

Problem 4: Electromagnetic Radiation

A particle with charge q moves on the xy-plane with position given by $(x = x_0 e^{-i\omega t}, y = y_0 e^{-i\omega t + i\pi/2})$, where x_0 and y_0 are the amplitudes of the motion and ω is the angular velocity.

(a) Calculate the dipole moment $\vec{p}$ associated with the particle in motion. [1.5 points]

(b) Calculate the B-field *in the far zone.* [2.5 points]

(c) Calculate the E-field *in the far zone.* [2.5 points]

(d) Evaluate the time-average of the Poynting vector. [2.5 points]

(e) Argue for, or explicitly calculate, the ω-dependence of the average power radiated due to the electric dipole. [1 point]

Useful Formulas:

$$\vec{B}(\vec{r}, t) = i\omega \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik \right) (\hat{r} \times \vec{p}(t))$$

$$\vec{E}(\vec{r}, t) = \frac{e^{ikr}}{4\pi\epsilon_0 r} \left[\left(\frac{1}{r^2} - \frac{ik}{r} \right) (3\hat{r}(\hat{r} \cdot \vec{p}(t)) - \vec{p}(t)) + k^2((\hat{r} \times \vec{p}(t)) \times \hat{r}) \right]$$

$$\vec{p}(t) = \int \vec{r'} \rho(\vec{r'}, t) d^3 r'$$

Problem 5: Electromagnetic Plane Waves

A plane wave with electric field strength E_0 traveling in vacuum is normally incident on a dielectric slab of finite thickness d and index of refraction η, as shown in the figure. Assume that the index of refraction of the slab is real (i.e., no absorption) and the medium on both sides of the slab is vacuum.

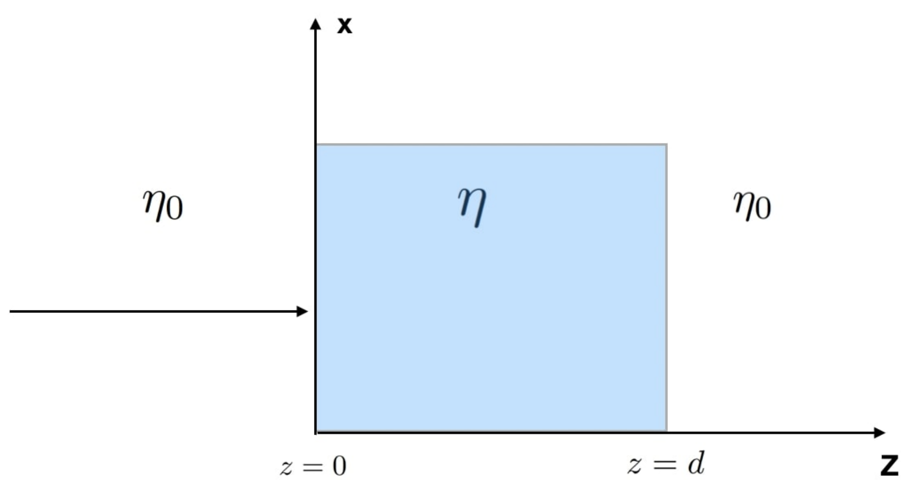
(a) Write down expressions for all the plane waves describing the electric and magnetic fields that need to be considered in this problem and their boundary conditions at $z = 0$ and $z = d$. [2 points]

(b) Consider first a situation where the slab is infinitely thick ($d \to \infty$). Calculate the reflection coefficient of the plane wave when it strikes the slab at the surface $z = 0$ from the left. [3 points]

(c) Now consider the situation where the slab has finite thickness d. What relationship between wavelength and thickness will give you minimum reflection at $z = 0$? What is the value of the reflection coefficient in this case? [1 point]

(d) Continue the case with the slab having finite thickness d. Calculate the reflection coefficient at the surface $z = d$. [2 points]

(e) Now consider a situation (not shown in the figure) where there is a slab with index of refraction η between $0 \leq z \leq d$ and another slab with index of refraction η in the interval $2d \leq z \leq \infty$, with vacuum everywhere else. A plane wave is incident on the slab at $z = 0$ from the left. Calculate the reflection coefficient at the surface $z = 2d$. [2 points]



Problem 6: Relativity

Consider a particle with rest mass m and charge q. Use relativistic expressions throughout this problem.

(a) Write down the relativistic expression for the particle's three-momentum $\vec{p}$ in terms of m and its three-velocity $\vec{v}$. You can keep the speed of light c in all formulae or set $c = 1$. [1 point]

(b) Write down the force law that gives $d\vec{p}/dt$ for the particle, where $\vec{p}$ is its 3-momentum, and the particle is in the presence of an electric field $\vec{E}$ and magnetic field $\vec{B}$. Assume that the velocity of the particle is $\vec{v}$. [1 point]

(c) Now consider the following scenario: the particle is at rest at the origin and an electric field $\vec{E} = E_0\hat{x}$ is present (the magnetic field is zero in this part). Assume that E_0 is a time-independent constant. Solve for the velocity $\vec{v}(t)$ and position $\vec{r}(t)$ of the particle, where t denotes time in the lab frame. From your expressions, find the velocity in the asymptotic large t limit and comment on its consistency with special relativity. Find the position $\vec{r}(t)$ in the small t limit, keeping to $\mathcal{O}(t^2)$ and comment on the non-relativistic limit. [2 points]

(d) Continue the case with the electric field $\vec{E} = E_0\hat{x}$, but this time assume that the initial velocity of the particle is $\vec{u} = u_0\hat{y}$. Determine the position $\vec{r}(t)$ of the particle as a function of time t in the lab-frame. [2 points]

(e) Let's switch off the electric field and instead consider the presence of a magnetic field $\vec{B} = B_0\hat{z}$. Assume that $\vec{B}$ is time-independent. The particle starts at the origin with initial velocity $\vec{u} = u_0\hat{x}$. Determine its velocity $\vec{v}(\vec{r}, t)$ and describe the motion in words. [3 points]

(f) Continue the case with the magnetic field $\vec{B} = B_0\hat{z}$ but now assume that the initial velocity of the particle is $\vec{u} = u_0\hat{x} + u_0\hat{z}$. Find $\vec{v}(\vec{r}, t)$ and describe the motion in words. [1 points]

Possibly useful integral:

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$$